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# Oblique and curved D-branes in IIB plane-wave string theory

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Abstract: Oblique $Dp$-branes in the maximally supersymmetric type-IIB plane-wave background are constructed in terms of boundary states, as well as from the open string point of view. These $Dp$-branes, whose existence was anticipated by Hikida and Yamaguchi from general supersymmetry arguments, have an isometry that is a subgroup of the diagonal SO(4) symmetry of the background. The oblique $D3$-brane is found to preserve four dynamical and four kinematical supersymmetries while the oblique $D5$-brane preserves one half of both the dynamical and kinematical supersymmetries. We also discuss the open-string boundary conditions for curved $D7$- and $D5$-branes, and analyze their supersymmetry.

Keywords: Superstrings and Heterotic Strings, D-branes, Penrose limit and pp-wave background.

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JHEP10(2003)052

## Contents



JHEP10(2003)052

## 1. Introduction

The study of string theory in $pp$-wave backgrounds provides an intriguing arena for extending the AdS/CFT correspondence to a situation in which excited string states can explicitly be related to composite Yang–Mills operators [1]. The maximally supersymmetric type-IIB plane-wave background can be obtained from $AdS_5 \times S^5$ by an infinite boost along a great circle of the five-sphere — the Penrose limit [2]. Some supersymmetric $D$-branes of the plane-wave string theory can be obtained by taking the Penrose limits of BPS branes of the $AdS_5 \times S^5$ theory. However, others originate from non-supersymmetric branes of the $AdS_5 \times S^5$ theory and only become supersymmetric in the Penrose limit.

The classification of $D$-branes in $pp$-wave backgrounds is incomplete. Two classes of $D$-branes have been explicitly constructed in the maximally supersymmetric case in [3]–[9], making use of a boundary state closed-string formalism as well as by describing the open-string states. Such branes are characterised by a gluing matrix, $M$, that describes how the left- and right-moving fermionic zero-modes of the closed string are related at the boundary. The orthogonal matrix $M$ is determined in terms of the geometry of the world-volume; if we label by $\mathcal{N}$ an orthonormal basis for the world-volume directions of the brane, then $M$ is given by

$$M = \prod_{J\in\mathcal{N}} \gamma^J \,, \tag{1.1}$$

where $\gamma^J_{a\dot{b}}$ are SO(8) gamma matrices, which couple inequivalent spinors (with dotted and undotted indices). The two classes of $D$-branes are:

**Class I.** The branes of this class are characterised by the condition

$$(\Pi M \Pi M)_{ab} = -\delta_{ab} \,, \tag{1.2}$$

where $\Pi = \gamma^1\gamma^2\gamma^3\gamma^4$.[1] All of these branes preserve eight dynamical and eight kinematical supersymmetries. In terms of the notation introduced in [5] where a $(r,s)$-brane has $r$ Neumann directions along $x^1, x^2, x^3, x^4$ and $s$ Neumann directions along $x^5, x^6, x^7, x^8$, this class includes branes of type $(r,s)$ with $|r-s| = 2$. We are here using the same conventions as in [8].[2]

**Class II.** In this case the gluing matrix satisfies

$$(\Pi M \Pi M)_{ab} = +\delta_{ab} \,. \tag{1.3}$$

The branes in this class that preserve some supersymmetry are $(0,0)$ (the euclidean $D$-instanton or lorentzian $D$-string) and $(0,4)$ or $(4,0)$. The latter are only consistent in the presence of world-volume flux. These $D$-branes preserve eight dynamical supersymmetries but do not preserve the kinematical supersymmetries.

A separate study of $D$-branes in the context of generalised string theory *pp*-wave backgrounds of [10] was performed in [11] (following [12]). In particular, backgrounds whose world-sheet theory has (dynamical) $(2,2)$ supersymmetry were considered, which have supersymmetric A- and B-type branes. The B-type $D$-branes (whose world-volume describes a holomorphic submanifold) have to satisfy that on the world-volume of the brane

$$W = \text{constant} \,, \tag{1.4}$$

where $W$ is the superpotential. The maximally supersymmetric plane-wave is a special background of this type with superpotential

$$W = i(z_1^2 + z_2^2 + z_3^2 + z_4^2) \,, \tag{1.5}$$

[1] We will also define $\widehat{\Pi} = \gamma^5\gamma^6\gamma^7\gamma^8$. A dotted spinor will be chosen to have positive SO(8) chirality so that $\left(\Pi\widehat{\Pi}\right)_{\dot{a}\dot{b}} = +\delta_{\dot{a}\dot{b}}$ and $\left(\Pi\widehat{\Pi}\right)_{ab} = -\delta_{ab}$.

[2] In particular, the notation $(r,s)$ applies both to the lorentzian signature $Dp$-branes (for which the light-cone $x^{\pm}$ directions are tangential to the world-volume) and to the instantonic brane (for which the light-cone $x^{\pm}$ directions are transverse to the world-volume).

where $z_i = x^i + ix^{i+4}$. The analysis of [11] therefore implies that any brane for which (1.5) is constant along the world-volume preserves at least some fraction of the $(2,2)$ supersymmetry. The additional A-type branes (which are special lagrangian submanifolds) are class II $(4,0)$ and $(0,4)$ branes with world-volume fluxes.

The solutions of the condition (1.4) include *oblique* $D$-branes, whose isometry group is a subgroup of the diagonal SO(4) subgroup of the background symmetry. These branes cannot be characterised in terms of the $(r,s)$-labels as above. The new $D$-branes found in [11] are the following:

i) Oblique $D3$-branes. In this case, the world-volume is parametrised e.g. by

$$z_1 = iz_2\,, \qquad z_3 = a\,, \qquad z_4 = b\,, \tag{1.6}$$

where $a$ and $b$ are complex constants. The two Neumann directions are therefore $x^1 - x^6$ and $x^2 + x^5$, and the corresponding gluing matrix is

$$M_3 = \frac{1}{2}(\gamma^1 - \gamma^6)(\gamma^2 + \gamma^5)\,. \tag{1.7}$$

It is easy to see that in this case

$$\Pi M_3 \Pi M_3 = \gamma^1\gamma^2\gamma^5\gamma^6\,. \tag{1.8}$$

Thus $M_3$ does not satisfy either (1.2) or (1.3).

ii) Oblique $D5$-branes. The complex parametrisation in this case e.g. takes the form

$$z_1 = iz_2\,, \qquad z_3 = iz_4\,, \tag{1.9}$$

so that $c = 0$ and the Neumann directions are given by

$$x^1 - x^6\,,\ x^2 + x^5\,,\ x^3 - x^8\,,\ x^4 + x^7\,. \tag{1.10}$$

The corresponding gluing matrix is

$$M_5 = \frac{1}{4}(\gamma^1 - \gamma^6)(\gamma^2 + \gamma^5)(\gamma^3 - \gamma^8)(\gamma^4 + \gamma^7)\,. \tag{1.11}$$

Again, by a direct computation one finds that

$$\Pi M_5 \Pi M_5 = \Pi\widehat{\Pi}\,. \tag{1.12}$$

Given our convention described above, the undotted component of (1.12) satisfies (1.2) above, and thus the $OD5$-brane is in fact a class I brane.

iii) Curved $D7$-brane. The world-volume in this case satisfies the constraint

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = c\,, \tag{1.13}$$

which describes a deformed conifold. An analogous curved $D5$-brane (which was not mentioned in [11]) will also be discussed.



JHEP10(2003)052

In this paper we will construct the boundary states of the oblique $D3$-brane and oblique $D5$-brane as well as their description in terms of open strings. We shall analyse how much supersymmetry these branes actually preserve. We will also discuss some aspects of the open string description of the curved $D7$-brane although the construction of the boundary state raises new issues which we will not address.

The guiding principle in the construction of the boundary states is the preservation of various supersymmetries. In the cases studied earlier, the condition that the boundary state is annihilated by half the dynamical light-cone supercharges was imposed,

$$\left(Q_{\dot{a}} + i\eta\, M_{\dot{a}\dot{b}}\, \tilde{Q}_{\dot{b}}\right) \|B, \eta\rangle\rangle = 0\,. \tag{1.14}$$

Together with the bosonic gluing conditions, this determined the gluing conditions for the fermionic modes.[3] In the context of the oblique branes, proceeding in this way can lead to inconsistent gluing conditions on the fermions. We will find that the oblique $D3$-brane preserves only a subalgebra of the full supersymmetry algebra. This is confirmed by an open string supersymmetry analysis, which will show that only four dynamical and four kinematical supersymmetries are preserved. In the boundary state analysis, one needs to project the condition in equation (1.14) onto the conserved supersymmetries in order to get consistent conditions. One of the main points of this paper is thus the construction of these D-branes that preserve less than half the supersymmetries from both the open string and closed string points of view. In the light of the BMN correspondence it would be of interest to understand the duals of these D-branes in the gauge theory although we will not address this issue here. By contrast to the oblique $D3$-brane, the oblique $D5$-brane turns out to be a class I brane which preserves eight dynamical and eight kinematical supersymmetries. As a result the projection on the supercharges in this case is trivial.

The outline of this paper is as follows. In section 2, the open string supersymmetries are analysed and the relevant projectors on the supersymmetries are determined. The open string mode expansions are found, and the corresponding one-loop diagrams are calculated. In section 3, the boundary states for the oblique $D3$-brane and $D5$-brane are constructed. Their tree-level cylinder diagrams are determined and shown to agree with the open string one-loop diagrams.[4] Section 4 presents the open string description of the curved $D7$-brane, as well as the analysis of the curved $D5$-brane.

## 2. The open-string point of view

Type-IIB superstring theory in the maximally supersymmetric plane-wave background [13, 14] is naturally formulated in light-cone gauge. In this gauge the 32 supersymmetries of

[3] The condition (1.14) is compatible with the $\mathrm{SO}(4) \times \mathrm{SO}(4)$ isometry of the background. Generalisations of this condition which explicitly break this symmetry will not be considered in this paper.

[4] The considerations of this paper apply equally well to branes with world-volumes of euclidean or lorentzian signature. In fact, as discussed in [8, 6], the calculations for a euclidean brane with $(p-1)$-dimensional world-volume are essentially identical to those of a lorentzian brane with $(p+1)$-dimensional world-volume, and we shall therefore refer to both as $D(p+1)$-branes. For simplicity of notation we shall usually consider lorentzian branes in the open string description and euclidean branes in the closed.

the background divide into sixteen nonlinearly realised 'kinematical' supersymmetries and sixteen linearly realised 'dynamical' supersymmetries. A subset of these supersymmetries is preserved by the oblique brane boundary conditions, as we will now see. The light-cone gauge plane-wave IIB action is given by (using the conventions as in [8])

$$I = \frac{1}{2\pi\alpha'}\int d\tau \int_0^\pi d\sigma \left[\partial_+ X^I \partial_- X^I - \widehat{m}^2 (X^I)^2 - (S\partial_+ S + \tilde{S}\partial_-\tilde{S} - 2\widehat{m} S\Pi\tilde{S})\right]. \tag{2.1}$$

The dynamical supersymmetry transformations of the coordinates are given by

$$\begin{aligned} \delta_\epsilon X^I &= \left[\epsilon^1\gamma^I S + \epsilon^2\gamma^I\tilde{S}\right] \\ \delta_\epsilon S^a &= \left[\partial_- X^I (\gamma^I)^{a\dot{b}}\epsilon^1_{\dot{b}} + \widehat{m} X^I (\Pi\gamma^I)^{a\dot{b}}\epsilon^2_{\dot{b}}\right] \\ \delta_\epsilon \tilde{S}^a &= \left[\partial_+ X^I (\gamma^I)^{a\dot{b}}\epsilon^2_{\dot{b}} - \widehat{m} X^I (\Pi\gamma^I)^{a\dot{b}}\epsilon^1_{\dot{b}}\right], \end{aligned} \tag{2.2}$$

while the kinematical supersymmetry transformations are

$$\begin{aligned} \delta_\kappa S &= \cos\widehat{m}\tau\,\kappa^1 + \sin\widehat{m}\tau\,\Pi\kappa^2 \\ \delta_\kappa \tilde{S} &= \cos\widehat{m}\tau\,\kappa^2 - \sin\widehat{m}\tau\,\Pi\kappa^1 . \end{aligned} \tag{2.3}$$

The fermionic coordinates of an open string with end-points on the brane satisfy the boundary conditions

$$S(\sigma,\tau)|_{\sigma=0,\pi} = M\tilde{S}(\sigma,\tau)|_{\sigma=0,\pi} . \tag{2.4}$$

The number of supersymmetries which are preserved on the $D$-brane can be determined by the requirement that the boundary conditions (2.4) (as well as the Dirichlet or Neumann boundary conditions on $X^I$) are invariant under supersymmetry transformations,

$$\delta S|_{\sigma=0,\pi} = M\delta\tilde{S}|_{\sigma=0,\pi} . \tag{2.5}$$

## 2.1 Dynamical supersymmetries

If $I$ labels a Dirichlet direction, $I \in \mathcal{D}$, then the supersymmetry variation of $X^I$ (2.2) must vanish at the boundary. Together with (2.4) and imposing

$$\epsilon^1 = -M\epsilon^2 , \tag{2.6}$$

leads to the condition on the gluing matrix $M$,

$$M^t\gamma^I M - \gamma^I = 0 , \quad I \in \mathcal{D} . \tag{2.7}$$

[Equivalently, we could have chosen $\epsilon_1 = M\epsilon_2$ with $M^t\gamma^I M + \gamma^I = 0$, but the end result is the same.]

Next we require that the fermionic boundary condition (2.4) is invariant under the supersymmetry variation. It follows directly from (2.2) that

$$0 = \delta_\epsilon (S - M\tilde{S})|_{\sigma=0,\pi} = -\left[(\partial_- X^I)\gamma^I M + (\partial_+ X^I) M\gamma^I\right]\epsilon^2|_{\sigma=0,\pi} + O(\widehat{m}) , \tag{2.8}$$



JHEP10(2003)052

where $O(\widehat{m})$ denotes terms that depend on the mass parameter, and we have used (2.6). If $I$ is a Dirichlet direction, $\partial_\tau X^I = \partial_+ X^I + \partial_- X^I = 0$ at the boundary, and the corresponding terms in the bracket of (2.8) vanish by (2.7). On the other hand, if $I$ labels a Neumann direction, $I \in \mathcal{N}$, the relevant boundary condition is $\partial_+ X^I - \partial_- X^I = \partial_\sigma X^I = 0$, and this leads to[5]

$$M^t \gamma^I M + \gamma^I = 0\,, \quad I \in \mathcal{N}\,. \tag{2.9}$$

The remaining ($\widehat{m}$-dependent) terms of (2.8) are then

$$\delta_\epsilon (S - M\tilde{S})|_{\sigma=0,\pi} = \widehat{m}\left(\sum_{J\in\mathcal{N}} (\Pi + M\Pi M)\gamma^J X^J + \sum_{I\in\mathcal{D}} (\Pi - M\Pi M)\gamma^I X^I\right)\epsilon^2 = 0\,. \tag{2.10}$$

If the $D$-brane is located at the origin in the Dirichlet directions the second term is zero. The condition that the first term (associated with Neumann directions) also vanishes is

$$(\mathbb{1} + \Pi M \Pi M)\,\gamma^J \epsilon^2 = 0\,, \tag{2.11}$$

for all $J \in \mathcal{N}$ and $\epsilon^1 = -M\epsilon^2$. For the following it is useful to define the projectors

$$P^\pm = \frac{1}{2}(1 \pm \Pi M \Pi M)\,. \tag{2.12}$$

Consider now the case of the oblique $D3$-brane, where $M = M_3$ (1.7), and (2.12) is

$$P_3^\pm = \frac{1}{2}\left(\mathbb{1} \pm \gamma^1\gamma^2\gamma^5\gamma^6\right), \tag{2.13}$$

as follows from (1.8). The condition (2.11) can be rewritten by commuting $\gamma^J$, for each Neumann direction $J$, to the left of the bracket. This results in the equation

$$P_3^- \epsilon_2 = \frac{1}{2}\left(\mathbb{1} - \gamma^1\gamma^2\gamma^5\gamma^6\right)\epsilon^2 = 0\,. \tag{2.14}$$

Thus if $\epsilon^2$ satisfies (2.14), the boundary condition is invariant under the corresponding supersymmetry transformation.

At first this conclusion only holds if the $OD3$ is located at the origin. However, the second term in (2.10) is

$$\sum_{I\in\mathcal{D}} X^I \Pi\left(\mathbb{1} - \gamma^1\gamma^2\gamma^5\gamma^6\right)\gamma^I\epsilon^2\,. \tag{2.15}$$

Since $\epsilon^2$ satisfies (2.14), it follows that the $OD3$-brane continues to be supersymmetric if it is placed at an arbitrary position along $x^3, x^4, x^7, x^8$. This is in agreement with the properties of the oblique $D3$-brane in [11], which permitted the brane to be at arbitrary values of $z_3 = x_3 + ix_7$ and $z_4 = x_4 + ix_8$. It is also in agreement with the boundary state analysis in the closed-string sector that will be considered in the next section.

[5]In reaching this conclusion we have assumed that the terms independent of $\widehat{m}$ have to vanish by themselves; for branes that satisfy standard Dirichlet or Neumann boundary conditions for the bosons, this is necessarily the case, given the structure of the $\widehat{m}$-dependent term, see (2.10) below. However, for example for the case of the $(4,0)$- and $(0,4)$-brane this assumption is not satisfied.



JHEP10(2003)052

For the case of the oblique $D5$-brane, $M = M_5$ (1.11), and $\Pi M_5 \Pi M_5 = \Pi\widehat{\Pi}$ as was already observed in (1.12). Since $\epsilon^2$ has a dotted index and $\gamma^J$ changes the chirality of the spinor, the first bracket in equation (2.11) has undotted indices. In our conventions, $\Pi\widehat{\Pi} = -\mathbb{1}$ for the undotted indices, and thus (2.11) is satisfied without any further constraint on $\epsilon_2$. It is also easy to see that the brane is confined to the origin (again in agreement with [11]). Both of these statements actually follow from the fact that the gluing matrix $M_5$ satisfies the class I condition (1.2) for the undotted indices.

### 2.1.1 A more general analysis

To analyse in general which gluing matrices $M$ will preserve a certain fraction of the dynamical supersymmetries, it is convenient to introduce, for each Neumann direction $X^J$, the matrix

$$K^J = \gamma^J \Pi M \Pi M \gamma^J \,, \tag{2.16}$$

so that $\Pi M \Pi M \gamma^J = \gamma^J K^I$. With this notation (2.11) becomes

$$\frac{1}{2}\left(\mathbb{1} + K^J\right)\epsilon_2 = 0\,. \tag{2.17}$$

A necessary condition for this to have non-trivial solutions (except for the case, when there are no Neumann directions, as is the case for the $(0,0)$-brane) is that

$$\mathbb{1} = K^J K^J = \gamma^J K^J K^J \gamma^J = (\Pi M \Pi M)^2\,. \tag{2.18}$$

The fraction of dynamical supersymmetry that is preserved is determined by the dimension of the joint kernel of the different projectors in (2.17).

Since $M$ is a product of an even number of gamma matrices, it follows from (2.16) that $K^I$ can be written as a sum of products of 0, 4 or 8 gamma matrices. For the usual class I branes, $K^J = -\mathbb{1}$ for all $J$, and the kernel of the projectors has dimension 8 trivially. On the other hand, for the oblique $D5$-brane, each $K^J = \Pi\widehat{\Pi}$, and thus again the kernel of the projectors has dimension 8. The situation is different for the class II branes where $K^J = \mathbb{1}$ for all $J$. The only supersymmetric $D$-branes are then the $D$-instanton (which does not have any Neumann directions) and the $(4,0)$ and $(0,4)$ branes, for which the above analysis does not apply because of their world-volume flux. In case of the (genuinely) oblique branes $K^J$ is a product of 4 gamma matrices and when $K^J$ is the same for all Neumann directions (as is the case for the $OD3$-brane), a quarter of the dynamical supersymmetries is preserved.

## 2.2 Kinematical supersymmetries

Equations (2.3), together with the boundary conditions, lead to

$$\kappa^1 = M\kappa^2, \qquad P^+\kappa^2 = \frac{1}{2}(\mathbb{1} + \Pi M \Pi M)\kappa^2 = 0\,. \tag{2.19}$$

Thus some kinematical supersymmetries are preserved if the dimension of the kernel of $\frac{1}{2}(\mathbb{1} + \Pi M \Pi M)$ is non-zero. For the $OD3$ case $\Pi M_3 \Pi M_3 = \gamma^1\gamma^2\gamma^5\gamma^6$, and the dimension of the kernel of $P_3^+$ is four. This implies that the $OD3$-brane preserves four kinematical supersymmetries, which is one quarter of the background kinematical supersymmetries. For the $OD5$ case $\Pi M_5 \Pi M_5 = \Pi\widehat{\Pi}$, so that $\frac{1}{2}(\mathbb{1} + \Pi M_5 \Pi M_5)$ vanishes trivially (since we have made the chirality choice $\Pi\widehat{\Pi} = -\mathbb{1}$ for undotted spinors, and (2.19) is again an undotted equation). Therefore, in this case eight kinematical supersymmetries are preserved, as usual for a class I brane.

### 2.3 Open string mode expansion for the $OD3$-brane

In order to be able to determine the various cylinder diagrams it is useful to describe the boundary condition (2.4) in terms of modes. (In the following we shall only consider the fermionic fields; the mode expansion for the bosonic fields are as usual. ) As is explained in [8], the fermionic fields have the mode expansion

$$\begin{aligned} S(\sigma,\tau) &= S_0' \cos(\widehat{m}\tau) + \Pi\tilde{S}_0' \sin(\widehat{m}\tau) + S_0 \cosh(\widehat{m}\sigma) + \Pi\tilde{S}_0 \sinh(\widehat{m}\sigma) + \\ &\quad + \sum_{n\neq 0} c_n \left[ S_n\, e^{-i(\widehat{\omega}_n\tau - n\sigma)} + \frac{i}{\widehat{m}}(\widehat{\omega}_n - n)\Pi\tilde{S}_n e^{-i(\widehat{\omega}_n\tau + n\sigma)} \right], \end{aligned} \tag{2.20}$$

$$\begin{aligned} \tilde{S}(\sigma,\tau) &= -\Pi S_0' \sin(\widehat{m}\tau) + \tilde{S}_0' \cos(\widehat{m}\tau) + \tilde{S}_0 \cosh(\widehat{m}\sigma) + \Pi S_0 \sinh(\widehat{m}\sigma) + \\ &\quad + \sum_{n\neq 0} c_n \left[ \tilde{S}_n e^{-i(\widehat{\omega}_n\tau + n\sigma)} - \frac{i}{\widehat{m}}(\widehat{\omega}_n - n)\Pi S_n e^{-i(\widehat{\omega}_n\tau - n\sigma)} \right], \end{aligned} \tag{2.21}$$

where $\widehat{\omega}_n = \sqrt{\widehat{m}^2 + n^2}$, and $\widehat{m}$ is the mass parameter in the associated open string light cone gauge.

Let us first analyse the open string both of whose endpoints lie on an oblique $D3$-brane. For the zero modes (2.4) with $M = M_3$ (1.7) implies

$$S_0' = M_3 \tilde{S}_0' \,, \qquad \tilde{S}_0' = -\Pi M_3 \Pi S_0' \,, \tag{2.22}$$

$$S_0 = M_3 \tilde{S}_0 \,, \qquad \tilde{S}_0 = \Pi M_3 \Pi S_0 \,. \tag{2.23}$$

Given (1.8), these equations are only consistent provided we project $S_0'$ and $S_0$ onto the components $P_3^- S_0'$, and $P_3^+ S_0$, where $P_3^\pm$ was defined in (2.13). Thus this open string has four zero modes $P_3^+ S_0$, and four zero modes $P_3^- S_0'$.

For the non-zero modes the boundary condition (2.4) with $M = M_3$ implies

$$S_n = \mathcal{N}_n \tilde{S}_n \,, \tag{2.24}$$

where $n \in \mathbb{Z}$ and

$$\mathcal{N}_n = \left( \mathbb{1} + i\, \frac{(\widehat{\omega}_n - n)}{\widehat{m}}\, M_3 \Pi \right)^{-1} \left( \mathbb{1} - i\, \frac{(\widehat{\omega}_n - n)}{\widehat{m}}\, \Pi M_3^t \right) M_3 \,. \tag{2.25}$$

By using that $\Pi^\dagger \Pi = \mathbb{1}$ and $M_3^\dagger = M_3^t = -M_3$, it follows directly that $\mathcal{N}_n$ is a unitary matrix,

$$\mathcal{N}_n \mathcal{N}_n^\dagger = \mathbb{1} \,. \tag{2.26}$$

In order to write $\mathcal{N}_n$ more explicitly, it is convenient to write (2.24) separately for the $P_3^\pm$ components. Since $P_3^\pm$ commutes with $M_3$ and $\Pi$,

$$[P_3^\pm, M_3] = [P_3^\pm, \Pi] = 0 \,, \tag{2.27}$$

(2.24) can be written as

$$P_3^{\pm} S_n = \mathcal{N}_n^{\pm}\, P_3^{\pm} \tilde{S}_n\,, \tag{2.28}$$

where

$$\mathcal{N}_n^{+} = \frac{1}{\widehat{\omega}_n}\left[n\mathbb{1} - i\widehat{m} M_3 \Pi\right] M_3\,, \quad \mathcal{N}_n^{-} = M_3\,. \tag{2.29}$$

Thus it follows that the minus components behave just as for a usual class I brane, while the behaviour of the plus components is as for a class II brane. Because of the four fermionic zero modes $P_3^{+} S_0$, the one loop (cylinder) diagram of this open string vanishes.

For a string stretching between a $OD3$-brane and its anti-brane $\overline{OD3}$ none of the zero modes above survives. As regards the non-zero modes, the conditions arising at $\sigma = 0$ are (2.28), while at $\sigma = \pi$, one finds

$$P_3^{\pm} S_n = -e^{2\pi i n}\, \widehat{\mathcal{N}}_n^{\pm}\, P_3^{\pm} \tilde{S}_n\,, \tag{2.30}$$

where now

$$\widehat{\mathcal{N}}_n^{+} = \frac{1}{\widehat{\omega}_n}\left[n\mathbb{1} + i\widehat{m} M_3 \Pi\right] M_3\,, \quad \widehat{\mathcal{N}}_n^{-} = M_3\,. \tag{2.31}$$

In order to solve (2.28) and (2.30) simultaneously, the minus components $P_3^{-} S_n$ must have $n \in \mathbb{Z} + \frac{1}{2}$, while two of the plus components $P_3^{+} S_n$ satisfy

$$\frac{n - i\widehat{m}}{n + i\widehat{m}} = -e^{2\pi i n}\,, \tag{2.32}$$

while the other two plus components satisfy

$$\frac{n + i\widehat{m}}{n - i\widehat{m}} = -e^{2\pi i n}\,. \tag{2.33}$$

In fact, the two components that satisfy (2.32) are the two eigenstates with eigenvalue $+1$ under the action of $\Pi$, while for the two components in (2.33) the eigenvalue of $\Pi$ is $-1$. The solution $n = 0$ is to be discarded in either case. As in [8] one can thus deduce that the open string one-loop diagram in this case equals

$$\mathcal{A}_{\overline{OD3},OD3} = \frac{\left(\hat{g}_4^{(\widehat{m})}(\tilde{q})\right)^2 \left(f_2^{(\widehat{m})}(\tilde{q})\right)^4}{\left(f_1^{(\widehat{m})}(\tilde{q})\right)^8}\,, \tag{2.34}$$

where the functions $f_i^{(\widehat{m})}$ and $\hat{g}_4^{(\widehat{m})}$ are those defined in [6, 8], and we have assumed, for simplicity, that both branes are located at the origin in the transverse space.

## 3. Boundary states for the oblique branes

We now turn to the description of these oblique $D$-branes in terms of boundary states. These are closed-string states that describe the coupling of the brane to arbitrary closed-string states. This is most simply formulated in the light-cone gauge with the $X^{\pm}$ directions transverse to the brane [15] — in other words, for branes with euclidean world-volumes, or instantonic branes. As argued in [6], the description of the supersymmetry conditions for a euclidean brane with $(p-1)$-dimensional world-volume is closely related to that of a lorentz signature $D(p+1)$-brane with longitudinal $X^{\pm}$. The dynamical supercharges (for which suitable superpositions are preserved by the boundary) are explicitly given as (see [8] for an explanation of the notation)

$$\begin{aligned}\sqrt{2p^+}\, Q_{\dot a} &= p_0^I \gamma^I_{\dot a b} S_0^b - m x_0^I \left(\gamma^I \Pi\right)_{\dot a b} \tilde S_0^b + \qquad (3.1)\\ &+ \sum_{n=1}^{\infty} \left( c_n \gamma^I_{\dot a b} (\alpha^I_{-n} S_n^b + \alpha^I_n S^b_{-n}) + \frac{im}{2\omega_n c_n} \left(\gamma^I \Pi\right)_{\dot a b} (\tilde\alpha^I_{-n} \tilde S_n^b - \tilde\alpha^I_n \tilde S^b_{-n}) \right),\\ \sqrt{2p^+}\, \tilde Q_{\dot a} &= p_0^I \gamma^I_{\dot a b} \tilde S_0^b + m x_0^I \left(\gamma^I \Pi\right)_{\dot a b} S_0^b + \qquad (3.2)\\ &+ \sum_{n=1}^{\infty} \left( (c_n \gamma^I_{\dot a b} (\tilde\alpha^I_{-n} \tilde S_n^b + \tilde\alpha^I_n \tilde S^b_{-n}) - \frac{im}{2\omega_n c_n} \left(\gamma^I \Pi\right)_{\dot a b} (\alpha^I_{-n} S_n^b - \alpha^I_n S^b_{-n}) \right).\end{aligned}$$

They satisfy the superalgebra

$$\{Q^+_{\dot a}, Q^-_{\dot b}\} = 2\,\delta_{\dot a\dot b}\, H + m\,(\gamma^{ij}\,\Pi)_{\dot a\dot b}\, J^{ij} + m\,(\gamma^{i'j'}\,\Pi)_{\dot a\dot b}\, J^{i'j'}\,, \qquad (3.3)$$

where $Q^{\pm}_{\dot a} = \frac{1}{\sqrt{2}}(Q_{\dot a} \pm i\tilde Q_{\dot a})$.

### 3.1 The $OD3$-brane

Let us begin by describing the oblique $D3$-brane. For definiteness, we consider the brane that has Neumann directions along $x^1 - x^6$ and $x^2 + x^5$, whose gluing matrix is given by $M_3$ in (1.7). Any other $OD3$-brane can be obtained from this by a $\mathrm{SO}(4)\times\mathrm{SO}(4)$ rotation.

The gluing conditions for the bosonic modes are obtained in a standard fashion as

$$\begin{aligned}\left(a^i_n - \tilde a^i_{-n}\right) \|D,\eta\,\rangle\!\rangle &= 0\,, \quad i\in\mathcal{D}\\ \left(a^i_n + \tilde a^i_{-n}\right) \|D,\eta\,\rangle\!\rangle &= 0\,, \quad i\in\mathcal{N}\,. \qquad (3.4)\end{aligned}$$

Here $a^i_n$ are the modes appropriate to an adapted orthogonal basis; for example, for the Neumann direction $i = 1-6$, $a^i_n = \alpha^1_n - \alpha^6_n$, where $\alpha^i_n$ are the modes of the bosonic field $X^i(z)$.

Given the analysis of the open string, the boundary states we are interested in should satisfy a projected supercharge condition of the form

$$P_3^+ \left( Q + i\eta\, M_3\, \tilde Q \right) \|OD3,\eta\,\rangle\!\rangle = 0\,. \qquad (3.5)$$

The strategy of the construction (as in [8]) is to determine the fermionic gluing conditions from the supercharge condition and the bosonic gluing conditions.

#### 3.1.1 The zero mode gluing conditions

We will first consider the zero mode part of the condition (3.5), i.e. the terms proportional to $p_0^I$ and $x_0^I$. If the brane is located at the origin in the transverse directions, the conditions for the Dirichlet directions (i.e. the terms proportional to $p_0^I$, where $I$ is a Dirichlet direction) are

$$P_3^+ \left(\gamma^I\, S_0 + i\eta\, M_3\, \gamma^I\, \tilde S_0\right) \|OD3,\eta\,\rangle\!\rangle = 0\,, \qquad (3.6)$$

where $I$ is a Dirichlet direction. Since $\gamma^I M_3 = M_3 \gamma^I$, (3.6) simplifies to

$$P_3^+ \gamma^I \left( S_0 + i\eta \, M_3 \, \tilde{S}_0 \right) \| OD3, \eta \rangle\rangle = 0 \,. \tag{3.7}$$

If $I$ is a Dirichlet direction other than $x^1 + x^6$ or $x^2 - x^5$ (i.e. $I = 3, 4, 7, 8$), then $\gamma^I$ commutes with $P_3^+$. Multiplying (3.7) by $\gamma^I$ then implies

$$P_3^+ \left( S_0 + i\eta \, M_3 \, \tilde{S}_0 \right) \| OD3, \eta \rangle\rangle = 0 \,. \tag{3.8}$$

On the other hand, if $I$ corresponds to the Dirichlet directions $x^1 + x^6$ or $x^2 - x^5$, then $P_3^+ \gamma^I = \gamma^I P_3^-$. Multiplying (3.7) by $\gamma^I$ then leads to

$$P_3^- \left( S_0 + i\eta \, M_3 \, \tilde{S}_0 \right) \| OD3, \eta \rangle\rangle = 0 \,. \tag{3.9}$$

Adding this to (3.8) it follows that the zero modes must satisfy the condition

$$\left( S_0 + i\eta \, M_3 \tilde{S}_0 \right) \| OD3, \eta \rangle\rangle = 0 \,. \tag{3.10}$$

We must now check whether (3.10) is compatible with the constraints that arise for the Neumann directions, for which $p_0^J = 0$ on the boundary state. The zero mode part of (3.5) (i.e. the constraints coming from the terms proportional to $x_0^J$ with $J$ a Neumann direction) implies

$$P_3^+ \left( M_3 \, \gamma^J \, \Pi \, S_0 + i\eta \, \gamma^J \, \Pi \, \tilde{S}_0 \right) \| OD3, \eta \rangle\rangle = 0 \,. \tag{3.11}$$

Using $M_3 \gamma^J = -\gamma^J M_3$ this can be rewritten as

$$- P_3^+ \, \gamma^J \, M_3 \, \Pi \left( S_0 - i\eta \, \Pi M_3^t \Pi \, \tilde{S}_0 \right) \| OD3, \eta \rangle\rangle = 0 \,. \tag{3.12}$$

Using the fact that $P_3^+ \gamma^J = \gamma^J P_3^-$ for Neumann directions, this condition is therefore equivalent to

$$P_3^- \left( S_0 - i\eta \, \Pi M_3^t \Pi \, \tilde{S}_0 \right) \| OD3, \eta \rangle\rangle = 0 \,, \tag{3.13}$$

which is compatible with (3.9) since

$$P_3^- M_3 = - P_3^- \Pi M_3^t \Pi \,. \tag{3.14}$$

Finally, we shall analyse whether the condition $x^I = 0$ when $I$ is a Dirichlet direction can be relaxed. The term that is proportional to $x^I$ in (3.5) is

$$P_3^+ \left( M_3 \, \gamma^I \, \Pi \, S_0 + i\eta \, \gamma^I \, \Pi \, \tilde{S}_0 \right) \| OD3, \eta \rangle\rangle = 0 \,. \tag{3.15}$$

This is the same as (3.11) but now $\gamma^I$ commutes with $M_3$, and therefore (3.15) becomes

$$P_3^+ \, \gamma^I \, M_3 \, \Pi \left( S_0 + i\eta \, \Pi M_3^t \Pi \, \tilde{S}_0 \right) \| OD3, \eta \rangle\rangle = 0 \,. \tag{3.16}$$

As before there are two cases to distinguish: if the Dirichlet direction corresponds to $I = 3, 4, 7, 8$ then $\gamma^I$ commutes with $P_3^+$. Multiplying (3.16) by $\gamma^I$ as well as $\Pi M_3^t$ then implies

$$P_3^+ \left( S_0 + i\eta \, \Pi M_3^t \Pi \, \tilde{S}_0 \right) \| OD3, \eta \rangle\rangle = 0 \,. \tag{3.17}$$

This is compatible with (3.8) since

$$P_3^+ M_3 = +P_3^+ \Pi M_3^t \Pi\,. \tag{3.18}$$

On the other hand, if $I$ is either $x^1 + x^6$ or $x^2 - x^5$, then (3.16) becomes

$$P_3^- \left(S_0 + i\eta\, \Pi M_3^t \Pi\, \tilde{S}_0\right) \| OD3, \eta\,\rangle\!\rangle = 0\,, \tag{3.19}$$

which is incompatible with (3.9) because of (3.14). It therefore follows that the oblique brane can be moved away from the origin along any of the $I = 3, 4, 7, 8$ directions (which are transverse to the $1-2-5-6$ plane) but that they are stuck at $x^1 = -x^6$ and $x^2 = x^5$.[6] This is in agreement with the open string analysis mentioned earlier, as well as the results of [11].

In order to characterise the fermionic ground state of the boundary state, it is convenient to rewrite (3.10) separately for the $P_3^+$ and $P_3^-$ components. Using

$$P_3^+ M_3 = P_3^+ \gamma^1\gamma^5\,, \qquad P_3^- M_3 = P_3^- \gamma^1\gamma^2\,, \tag{3.20}$$

it follows that

$$P_3^+ \left( S_0 + i\eta\, \gamma^1\gamma^5\, \tilde{S}_0 \right) \| OD3, \eta\,\rangle\!\rangle = 0\,, \tag{3.21}$$

and

$$P_3^- \left( S_0 + i\eta\, \gamma^1\gamma^2\, \tilde{S}_0 \right) \| OD3, \eta\,\rangle\!\rangle = 0\,. \tag{3.22}$$

### 3.1.2 Gluing conditions for the non-zero modes

For each Dirichlet direction (labeled by $I$) the gluing conditions that follow from (3.5) after substituting $\tilde{\alpha}_n^I = \alpha_{-n}^I$ (from (3.4)) and equating coefficients of $\alpha_n^I$ are

$$P_3^+ \gamma^I \left[ \left( \mathbb{1} + \frac{m\eta}{2\omega_n c_n^2} M_3 \Pi \right) S_n + i\eta \left( \mathbb{1} - \frac{m\eta}{2\omega_n c_n^2} \Pi M_3^t \right) M_3\, \tilde{S}_{-n} \right] \| OD3, \eta\,\rangle\!\rangle = 0\,. \tag{3.23}$$

For each Neumann direction (labeled by $J$) the gluing conditions that follow from (3.5) after substituting $\tilde{\alpha}_n^I = -\alpha_{-n}^I$ are

$$P_3^+ \gamma^J \left[ \left( \mathbb{1} - \frac{m\eta}{2\omega_n c_n^2} M_3 \Pi \right) S_n + i\eta \left( \mathbb{1} + \frac{m\eta}{2\omega_n c_n^2} \Pi M_3^t \right) M_3\, \tilde{S}_{-n} \right] \| OD3, \eta\,\rangle\!\rangle = 0\,. \tag{3.24}$$

By considering the two types of Dirichlet directions (i.e. $I = 3, 4, 7, 8$ or $I = 1+6, 2-5$), the same arguments as in the previous subsection show that (3.23) is equivalent to

$$\left[ \left( \mathbb{1} + \frac{m\eta}{2\omega_n c_n^2} M_3 \Pi \right) S_n + i\eta \left( \mathbb{1} - \frac{m\eta}{2\omega_n c_n^2} \Pi M_3^t \right) M_3\, \tilde{S}_{-n} \right] \| OD3, \eta\,\rangle\!\rangle = 0\,. \tag{3.25}$$

On the other hand, as with the zero modes, the Neumann condition (3.24) is equivalent to

$$P_3^- \left[ \left( \mathbb{1} - \frac{m\eta}{2\omega_n c_n^2} M_3 \Pi \right) S_n + i\eta \left( \mathbb{1} + \frac{m\eta}{2\omega_n c_n^2} \Pi M_3^t \right) M_3\, \tilde{S}_{-n} \right] \| OD3, \eta\,\rangle\!\rangle = 0\,. \tag{3.26}$$

[6] Using the isometry of the background, one can obviously move these branes away from the origin. However, the resulting branes then become explicitly time-dependent [7].

JHEP10(2003)052

Using (3.14) it follows that

$$P_3^- \left(S_n + i\eta\, M_3\, \tilde{S}_{-n}\right) \|OD3, \eta\,\rangle\!\rangle = P_3^- \left(S_n + i\eta\, \gamma^1\gamma^2\, \tilde{S}_{-n}\right) \|OD3, \eta\,\rangle\!\rangle = 0\,. \tag{3.27}$$

Thus the four components of $P_3^- S_n$ are related to $P_3^- \tilde{S}_{-n}$ by conventional gluing conditions of the same type as those of a class I $(2,0)$-brane.

As regards the $P_3^+$ components, the $P_3^+$ projection of (3.25) can be rewritten as

$$P_3^+ \left[S_n + i\eta \frac{1}{n} \left(\omega_n \mathbb{1} - m\eta\, M_3 \Pi\right) M_3\, \tilde{S}_{-n}\right] \|OD3, \eta\,\rangle\!\rangle = 0\,. \tag{3.28}$$

This relates the four components of $P^+ S_n$ to $P^+ \tilde{S}_{-n}$ as for a class II brane [8]. Finally, it may be worthwhile pointing out that these gluing conditions can be summarised in terms of the fields as

$$\left(S(\sigma,\tau) + i\eta\, M_3\, \tilde{S}(\sigma,\tau)\right)\Big|_{\tau=0} \|OD3, \eta\,\rangle\!\rangle = 0\,. \tag{3.29}$$

#### 3.1.3 The cylinder diagrams involving the $OD3$-brane

In section 2.3 the one-loop diagrams for the open string stretching between two $OD3$-branes, and the open string between an $OD3$-brane and its anti-brane were calculated. In particular, it was found that the former diagram vanished, while the latter was given by (2.34). We want to show now that these diagrams are correctly reproduced by the above boundary states, using the open closed duality relation. (See [6, 8] for a detailed explanation of this relation.)

The contribution coming from the non-zero modes of the boundary states is straightforward, but the analysis of the zero-modes requires some care. Since the gluing conditions do not mix the plus and minus components[7] the total contribution coming from the zero-modes is simply the product of the two contributions from plus- and minus-components, respectively. The $OD3$-brane satisfies the class I gluing condition (3.22) for the minus-components, and their contribution is therefore $(1 \pm q^m)^2$, where the sign depends on whether we consider the overlap between the brane and the brane, or the brane and the anti-brane [6]. In order to understand the contribution from the plus components, we rewrite the condition (3.21) in terms of the $\theta$ and $\bar{\theta}$ modes (as in [8]) as

$$P_3^+ \left(\theta - \eta\, \gamma^1\gamma^5\, \bar{\theta}\right) \|OD3, \eta\,\rangle\!\rangle^{(0)} = 0\,, \tag{3.30}$$

where $\|OD3, \eta\,\rangle\!\rangle^{(0)}$ denotes the zero-mode component of the boundary state. Since $\gamma^1\gamma^5$ anticommutes with $\Pi$, (3.21) relates $\theta_L$ to $\bar{\theta}_R$, and vice versa,

$$P_3^+ \left(\theta_L - \eta\, \gamma^1\gamma^5\, \bar{\theta}_R\right) \|OD3, \eta\,\rangle\!\rangle^{(0)} = 0 \tag{3.31}$$

$$P_3^+ \left(\theta_R - \eta\, \gamma^1\gamma^5\, \bar{\theta}_L\right) \|OD3, \eta\,\rangle\!\rangle^{(0)} = 0\,. \tag{3.32}$$

The operators that appear in the first line are both annihilation operators, while those in the second line are both creation operators. The boundary state is therefore created

[7] Here 'plus' and 'minus' refers to the eigenvalue under the action of $\Pi M_3 \Pi M_3$.



JHEP10(2003)052

from the ground state of the Fock space by the action of the creation operators of the second line. Since there are two such operators, each of which raises the energy by $m/2$, the resulting state has energy $m$. Since this state has definite eigenvalue under the closed string hamiltonian, the overlap between two brane states (i.e. two states with the same value for $\eta$) vanishes, while the overlap between the brane state and its anti-brane state is proportional to $q^m$. Together with the contribution from the non-zero modes, the whole cylinder amplitude between brane and brane vanishes, while the overlap between a brane and its anti-brane is

$$\tilde{\mathcal{A}}_{\bar{D}3_o,D3_o} = \frac{\left(g_2^{(m)}(q)\right)^2 \left(f_2^{(m)}(q)\right)^4}{\left(f_1^{(m)}(q)\right)^8}\,, \tag{3.33}$$

where $g_2^{(m)}$ is the function defined in [8]. Under the modular transformation $t \mapsto \tilde{t} = 1/t$, this reproduces indeed (2.34).

### 3.2 The $OD5$-brane

Next let us consider the $OD5$-brane with Neumann directions

$$x^1 - x^6\,,\; x^2 + x^5\,,\; x^3 - x^8\,,\; x^4 + x^7\,, \tag{3.34}$$

whose gluing matrix $M_5$ was already given in (1.11). A general oblique $D5$-brane can be obtained from the above by rotations in $\mathrm{SO}(4) \times \mathrm{SO}(4)$.

As we have discussed before, the $OD5$-brane behaves in fact as a class I brane, and thus the construction of the boundary states is as described in [6, 3]. In particular, the gluing conditions for the fermions are simply

$$\left(S_n + i\eta M_5 \tilde{S}_{-n}\right) \| OD5, \eta\,\rangle\!\rangle = 0\,, \tag{3.35}$$

and it is easy to see that, together with the gluing conditions for the bosons, this leads to

$$\left(Q + i\eta M_5 \tilde{Q}\right) \| OD5, \eta\,\rangle\!\rangle = 0\,. \tag{3.36}$$

Given that the oblique $D5$-brane is in fact a class I brane, one may wonder whether it is in fact related by a rotation to one of the standard class I branes discussed before; this will be discussed in the next subsection.

#### 3.2.1 Rotation symmetries

The maximally supersymmetric background is invariant under the $\mathrm{SO}(4)\times\mathrm{SO}(4)$ subgroup of the transverse $\mathrm{SO}(8)$ rotation group. The image of any supersymmetric brane under a rotation $R$ in $\mathrm{SO}(4) \times \mathrm{SO}(4)$ therefore describes another supersymmetric $D$-brane of the same kind. Indeed, under any rotation $R$, the brane characterised by $M$ is mapped to a brane described by $\widehat{M} = R^t M R$. If $R$ is an element in $\mathrm{SO}(4) \times \mathrm{SO}(4)$, $R$ commutes with $\Pi$, and hence

$$\Pi \widehat{M} \Pi \widehat{M} = \Pi\, R^t M R\, \Pi\, R^t M R = R^t\, \Pi M \Pi M\, R\,. \tag{3.37}$$

Thus if $M$ is of class I, so is $\widehat{M}$, and similarly for class II and oblique branes.

While elements in SO(4) $\times$ SO(4) map branes of class I into themselves, they are not the only transformations with this property. Suppose $R$ is an element in SO(8) that is not in SO(4) $\times$ SO(4). By composing $R$ with elements in SO(4) $\times$ SO(4), we may, without loss of generality assume, that $R$ has the property that

$$\Pi R = R^t \Pi\,. \tag{3.38}$$

The elements of SO(8) with this property define the coset SO(8)/(SO(4) $\times$ SO(4)).

The condition that $\widehat{M} = R^t M R$ defines again a class I brane (provided that $M$ does) then yields the constraint

$$(R^t)^2 M R^2 = M\,. \tag{3.39}$$

Since rotations that preserve the world-volume of a brane do not map different branes into one another, we may furthermore restrict attention to those rotations that are of the form $R = \prod \exp(\frac{\theta_{ij}}{2}\gamma^i\gamma^j)$ where $i \in \mathcal{N}$ and $j \in \mathcal{D}$. Then $MR = R^t M$, and (3.39) becomes

$$R^4 = \mathbb{1}\,. \tag{3.40}$$

Since the class I condition (1.2) only refers to the undotted indices, (3.40) only has to hold for them.

For a given brane of class I, there are only finitely many rotations that satisfy these constraints. For example, we can take $R$ to be of the form

$$R = \exp\left(\frac{\theta_1}{2}\gamma^{i_1}\gamma^{j_1}\right)\exp\left(\frac{\theta_2}{2}\gamma^{i_2}\gamma^{j_2}\right)\,, \tag{3.41}$$

where $i_1 \neq i_2 \in \mathcal{N}$ and $j_1 \neq j_2 \in \mathcal{D}$. The condition (3.40) is then obviously satisfied if $\theta_1 = \theta_2 = \pi$; this corresponds to a parity transformation. However, (3.40) is also satisfied if $\theta_1 = \theta_2 = \pi/2$. The resulting transformation then maps a class I $(r,s)$-brane into a class I $(s,r)$-brane.

Let us now return to the question of whether the oblique $D5$-brane can be obtained from a class I brane by a rotation. Consider the class I $(3,1)$-brane with $M = \gamma^1\gamma^2\gamma^3\gamma^7$, and define the rotation group element $R$ by

$$R = R^{(1,6,-)}R^{(2,5,+)}R^{(3,8,-)}R^{(4,7,-)}\,, \tag{3.42}$$

where for $m < n$

$$R^{(m,n,\pm)} = \frac{1}{\sqrt{4+2\sqrt{2}}}\left((1+\sqrt{2})\,\mathbb{1} \pm \gamma^m\gamma^n\right) = \cos(\theta/2)\,\mathbb{1} \pm \sin(\theta/2)\,\gamma^1\gamma^6\,, \tag{3.43}$$

with $\theta = \pi/4$. It is easy to check that

$$R^t\gamma^1 R = \frac{1}{\sqrt{2}}(\gamma^1 - \gamma^6)\,, \tag{3.44}$$

and

$$R^t\gamma^6 R = \frac{1}{\sqrt{2}}(\gamma^1 + \gamma^6)\,. \tag{3.45}$$

Similarly one analyses the other directions. Overall, this rotation therefore transforms $M$ into

$$M_5 = R^t M R\,. \tag{3.46}$$

$R$ is of the form discussed before and satisfies

$$R^4 = -\Pi\widehat{\Pi}\,. \tag{3.47}$$

This equation reduces to $R^4 = \mathbb{1}$ for the undotted indices, and thus satisfies the above conditions.

It follows from (3.43) that $R$ describes a rotation by $\pi/4$ in each of the four planes $x^1 - x^6$, $x^2 - x^5$, $x^3 - x^8$ and $x^4 - x^7$. In particular, $R$ therefore maps the $(3,1)$-brane extended along $x^1, x^2, x^3, x^7$ to the oblique $D5$-brane whose world-volume extends along the directions (3.34).

### 3.3 The projected superalgebra

The different supersymmetric $D$-branes that we have considered above preserve linear combinations of supercharges of the form $(Q + iM\tilde{Q})$, or for the case of the $OD3$-brane, some projection of this combination (3.5). One may thus ask which subalgebra of the supersymmetry algebra is generated by these generators. If we define $q = Q + iM\tilde{Q}$, one finds

$$\{q_{\dot a}, q_{\dot b}\} = 2(N-\tilde{N})\delta_{\dot a\dot b} - \left[\frac{m}{2}\left((\gamma^{ij}\Pi M^t)_{\dot a\dot b}J^{ij} + (\gamma^{i'j'}\Pi M^t)_{\dot a\dot b}J^{i'j'}\right) + (\dot a \leftrightarrow \dot b)\right], \tag{3.48}$$

where $i,j = 1,\ldots,4$, $i',j' = 5,\ldots,8$, and $N, \tilde{N}$ are the left and right-moving number operators; their difference always vanishes on the boundary state since it imposes the level-matching condition.[8] For class I branes, the rotation generators that appear in the second line involve rotations that either act on the world-volume directions of the corresponding brane, or on the space transverse to the world-volume. In either case, this is a symmetry of the $D$-brane since the class I branes are required to be at the origin in the transverse space. For example, for the case of the class I $(2,0)$-brane with $M = \gamma^1\gamma^2$, one finds

$$\{q_{\dot a}, q_{\dot b}\} = 2(N-\tilde{N})\delta_{\dot a\dot b} + m\left[\delta_{\dot a\dot b}J^{34} - \Pi_{\dot a\dot b}J^{12} - \left(\gamma^{i'j'}\gamma^3\gamma^4\right)_{\dot a\dot b}J^{i'j'}\right]. \tag{3.49}$$

On the other hand, for the class II $D$-instanton, the terms in the second line of (3.48) vanish; this is as expected, since the $D$-instanton can be moved to any position in the transverse space, and is therefore, in general, not invariant under any rotation.

For the case of the oblique $D5$-brane described by the matrix $M$ with

$$M = \frac{1}{4}\prod_{i=1}^{4}(\gamma^i \pm \gamma^{\hat\imath})\,, \tag{3.50}$$

where for each $i$, $\hat\imath \in \{5,6,7,8\}$ and $\hat\imath \neq \hat\jmath$ if $i \neq j$, one finds

$$\{q_{\dot a}, q_{\dot b}\} = 2(N-\tilde{N})\delta_{\dot a\dot b} + \frac{m}{2}\left(\gamma^{ji}\Pi M + M\Pi\gamma^{ij}\right)_{\dot a\dot b}\left(J^{ij} \pm J^{\hat\imath\hat\jmath}\right). \tag{3.51}$$

[8] Terms such as $N - \tilde{N}$ that vanish on physical states have been ignored in [14].

It is again clear that the rotations that appear on the right hand side leave the world-volume of the $OD5$-brane invariant.

Finally, for the case of the $OD3$-brane, the relevant generator of the supersymmetry algebra is not just $q$ (with $M = M_3$), but rather $P_3^+ q$. The relevant subalgebra of the supersymmetry algebra is then

$$\{P_3^+ q_{\dot a}, P_3^+ q_{\dot b}\} = 2(N - \tilde N)(P_3^+)_{\dot a\dot b} + m(P_3^+\gamma^{12}\Pi M_3)_{\dot a\dot b}(J^{12} - J^{56})\,. \tag{3.52}$$

The rotations that appear on the right hand side of (3.52) leave the world-volume of the $OD3$-brane invariant and act trivially on the transverse space. This is again as expected since the $OD3$-brane can be moved along the transverse directions $x^I$ with $I = 3, 4, 7, 8$, and thus is in general not invariant under any rotation of these directions.

### 3.4 Comparison with results based on $(2, 2)$ supersymmetry

Finally, we explain how our results tie in with the results of [11]. In [11] only the $(2, 2)$ world-sheet supersymmetry of the background was considered, and thus it was only assumed that there are two complex Killing spinors (that were denoted by $|0\rangle$ and $|\tilde 0\rangle$ in [11]). In terms of our description, these two states are the two spinor states that transform in an irreducible 2-dimensional representation of the diagonal SO(4) subgroup of the background symmetry (see [8] for more details). Let us denote by $\psi$ and $\chi$ the two left-moving components, and by $\tilde\psi$ and $\tilde\chi$ their right-moving counterparts. (In terms of the notation of [8], $\chi$ and $\tilde\chi$ are the 'top', and $\psi$ and $\tilde\psi$ the 'bottom' states.) We now propose that the relation between the two sets of states is given by

$$|0\rangle = \psi - \tilde\psi + i(\chi + \tilde\chi)\,, \tag{3.53}$$

$$|\tilde 0\rangle = \chi - \tilde\chi + i(\psi + \tilde\psi)\,. \tag{3.54}$$

The main evidence for this proposal is that it correctly reproduces the unbroken supersymmetries for the different branes that were considered by [11]. For example, it was shown in [11] that the supersymmetries that are preserved by the $OD3$-brane are proportional to $|\tilde 0\rangle$. Given the above identification, this means that the preserved supersymmetries should be proportional to $\chi - \tilde\chi$ and $\psi + \tilde\psi$. This agrees with our analysis above since the preserved dynamical supersymmetry generators include the 'top' and 'bottom' component of $Q + iM_3\tilde Q$. In fact, on the space spanned by the top and bottom component $M_3$ takes the form

$$M_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{3.55}$$

and thus we have $\psi = \tilde\psi$ and $\chi = -\tilde\chi$ at the boundary.

Similarly, for the case of the $OD5$-brane, the unbroken supersymmetries of [11] are $\alpha|0\rangle + \zeta|\tilde 0\rangle$, where $\alpha = -i\zeta^*$. The surviving supersymmetries are thus

$$2\zeta_1(\chi + i\tilde\psi) - 2\zeta_2(\psi + i\tilde\chi)\,. \tag{3.56}$$

On the space spanned by the top and bottom component $M_5$ takes the form

$$M_5 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.57}$$

and thus we have $\psi = i\tilde\psi$ and $\chi = i\tilde\chi$ at the boundary, in agreement with (3.56) above.



JHEP10(2003)052

## 4. The curved $D7$-brane

The branes we have discussed so far all have world-volumes that are flat hyperplanes in the transverse space. As was explained in the introduction, one expects on general grounds that any world-volume along which the superpotential (1.5) is constant should define a supersymmetric brane. The simplest example is obviously the brane whose world-volume is *defined* by (1.4) with $W$ given by (1.5). In terms of real coordinates, this world-volume is characterised by the two equations[9]

$$\sum_{i=1}^{4}(x^i x^i - x^{i+4}x^{i+4}) = c_1\,, \qquad \text{and} \qquad \sum_{i=1}^{4} x^i x^{i+4} = c_2\,. \tag{4.1}$$

Since there are two 'Dirichlet' directions (whose orientation depends on the position on the world-volume of the brane), the corresponding brane is a $D7$-brane [11]. In this section we want to analyse the open string whose endpoint lies on this hypersurface. In particular, we shall show that, together with suitable boundary conditions for the fermions, this boundary condition is invariant under two (dynamical) supersymmetry transformations.

The first step of the construction is to identify the matrix $M$ that describes the combination of supersymmetries that should be preserved by the brane. So far, we have always taken $M$ to be the product of the $\gamma$-matrices associated to the Neumann directions, but we could have equally taken $M$ to be the product of the $\gamma$-matrices associated to the Dirichlet directions. For the problem at hand this is more convenient, and we therefore define

$$M_7 = \frac{1}{x^I x^I}\left(x^i\gamma^i - x^{i+4}\gamma^{i+4}\right)\left(x^i\gamma^{i+4} + x^{i+4}\gamma^i\right). \tag{4.2}$$

It is not difficult to check that $M$ is orthogonal, and that

$$M^t = -M\,. \tag{4.3}$$

For the following analysis it is convenient to work with a more explicit description of the spinor representations. Let us write the $\gamma$-matrices in terms of fermionic creation and annihilation operators,

$$\gamma^i = b^i + (b^i)^\dagger\,, \tag{4.4}$$

$$\gamma^{i+4} = i\left(b^i - (b^i)^\dagger\right)\,, \tag{4.5}$$

where $\{b^{\dagger\, i}, b^j\} = \delta^{ij}$. The top and bottom components of a spinor (that will be denoted by $\uparrow$ and $\downarrow$ in the following) are the states that are annihilated by all $(b^i)^\dagger$ or all $b^i$, respectively. These are the states with weights $(+,+,+,+)$ and $(-,-,-,-)$. The dotted spinor states have an even number of $b$ excitations and thus contain the top and bottom components as well as six other components of the form $(+,+,-,-)$, together with permutations of $+$ and $-$. Undotted spinor states are created from these states by the action of an odd number

[9] In the following we shall always choose the convention that $i,j$ run from $1,\ldots 4$, while $I,J$ run from $1,\ldots,8$.

of creation or annihilation operators. An undotted spinor therefore consists of two types of states: one of these has weights $(+,+,+,-)$ together with three other permutations, and can be written as $|i\uparrow\rangle$ where $i$ labels the four distinct states. The other type of state has $(-,-,-,+)$ and permutations with one $+$, and can be written as $|i\downarrow\rangle$. It will prove convenient to decompose the SO(8) spinors into these two types of components. For example, we will write the eight components of $S$ as $S=(S^{i\uparrow},S^{i\downarrow})$, and likewise for $\tilde{S}$. Note that $\{S^{i\uparrow},S^{j\downarrow}\}=\delta^{ij}$ and $\{S^{i\uparrow},S^{j\uparrow}\}=0$. It is also convenient to introduce the following complex combinations of left- and right-moving fermionic fields

$$\mathcal{T}^a = \tilde{S}^a + iS^a\,, \tag{4.6}$$
$$\bar{\mathcal{T}}^a = \tilde{S}^a - iS^a\,, \tag{4.7}$$

where $a=i\uparrow$ or $a=i\downarrow$. Finally, for the bosonic coordinates we introduce the complex coordinates[10]

$$Z^i = X^i + iX^{i+4}\,, \qquad \bar{Z}^i = X^i - iX^{i+4}\,. \tag{4.8}$$

In these coordinates, $M_7$ then takes the form

$$M_7 = -i\left[b^i,(b^\dagger)^j\right]\frac{Z^j\bar{Z}^i}{|Z|^2}\,. \tag{4.9}$$

It is easy to see that $M_7$ acts on the top- and bottom components of the (dotted) spinors precisely as $M_3$ in (3.55); this is in agreement with the result of [11]. Under the dynamical supersymmetry transformations associated to $\epsilon_2=\uparrow$ or $\epsilon_2=\downarrow$ with $\epsilon_1=-M_7\epsilon_2$ the fields then transform as

$$\begin{aligned} \delta_\uparrow Z^i &= 0 & \delta_\downarrow Z^i &= 2\,\mathcal{T}^{i\uparrow} \\ \delta_\uparrow \bar{Z}^i &= 2\,\bar{\mathcal{T}}^{i\downarrow} & \delta_\downarrow \bar{Z}^i &= 0\,, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} \delta_\uparrow \mathcal{T}^{i\uparrow} &= (\partial_+ + \partial_-)Z^i & \delta_\downarrow \mathcal{T}^{i\uparrow} &= 0 \\ \delta_\uparrow \bar{\mathcal{T}}^{i\uparrow} &= (\partial_+ - \partial_-)Z^i & \delta_\downarrow \bar{\mathcal{T}}^{i\uparrow} &= -2i\widehat{m}\bar{Z}^i \\ \delta_\uparrow \mathcal{T}^{i\downarrow} &= 2i\widehat{m}Z^i & \delta_\downarrow \mathcal{T}^{i\downarrow} &= (\partial_+ - \partial_-)\bar{Z}^i \\ \delta_\uparrow \bar{\mathcal{T}}^{i\downarrow} &= 0 & \delta_\downarrow \bar{\mathcal{T}}^{i\downarrow} &= (\partial_+ + \partial_-)\bar{Z}^i\,, \end{aligned} \tag{4.11}$$

as follows directly from (2.2).

### 4.1 The supersymmetry algebra

The above transformation formulae must obviously respect the relations that arise from the supersymmetry algebra. In particular, one easily sees that

$$\delta_\uparrow\delta_\uparrow \equiv 0 \equiv \delta_\downarrow\delta_\downarrow\,. \tag{4.12}$$

Furthermore, the anti-commutator of $\delta_\uparrow$ and $\delta_\downarrow$ must satisfy

$$\{\delta_\uparrow,\delta_\downarrow\} \equiv 2(\partial_+ + \partial_-) = 2\partial_\tau\,. \tag{4.13}$$

[10] We reserve upper-case letters for fields, and use lower-case letters to describe the geometry.

It is easy to see that (4.13) holds when acting on the bosonic fields, or $\mathcal{T}^{i\uparrow}$ and $\bar{\mathcal{T}}^{i\downarrow}$, but the analysis for the other two fermionic components requires more care:

$$\begin{aligned}\{\delta_\uparrow, \delta_\downarrow\}\bar{\mathcal{T}}^{i\uparrow} &= \delta_\downarrow(\partial_+ - \partial_-)Z^i - 2i\widehat{m}\delta_\uparrow \bar{Z}^i &(4.14)\\ &= 2\left[(\partial_+ - \partial_-)\mathcal{T}^{i\uparrow} - 2i\widehat{m}\bar{\mathcal{T}}^{i\downarrow}\right] &(4.15)\\ &= 2\left[(\partial_+ + \partial_-)\bar{\mathcal{T}}^{i\uparrow} + 2i\partial_+ S^{i\uparrow} - 2\partial_-\tilde{S}^{i\uparrow} - 2i\widehat{m}\bar{\mathcal{T}}^{i\downarrow}\right]. &(4.16)\end{aligned}$$

The last three terms in the bracket vanish by virtue of the equations of motion

$$\begin{aligned}\partial_+ S^{i\uparrow} &= \widehat{m}\Pi\tilde{S}^{i\uparrow} = \widehat{m}\tilde{S}^{i\downarrow} &(4.17)\\ \partial_-\tilde{S}^{i\uparrow} &= -\widehat{m}\Pi S^{i\uparrow} = -\widehat{m}S^{i\downarrow}\,, &(4.18)\end{aligned}$$

which imply, in particular, that

$$2i\partial_+ S^{i\uparrow} - 2\partial_-\tilde{S}^{i\uparrow} = 2i\widehat{m}\left(\tilde{S}^{i\downarrow} - iS^{i\downarrow}\right) = 2i\widehat{m}\bar{\mathcal{T}}^{i\downarrow}\,. \tag{4.19}$$

The analysis for the $\mathcal{T}^{i\downarrow}$ components is similar.

### 4.2 The supersymmetry analysis

We are now ready to analyse the supersymmetry of the boundary conditions that characterise the curved $D7$-brane. For the bosonic fields the 'Dirichlet' boundary conditions are

$$Z^i Z^i = c\,, \qquad \text{and} \qquad \bar{Z}^i\bar{Z}^i = \bar{c}\,, \tag{4.20}$$

where $c$ is a complex constant (and $\bar{c}$ its complex conjugate). Furthermore, we postulate that the fermionic boundary conditions are given by

$$S = -M_7\tilde{S}\,, \tag{4.21}$$

where $M_7$ is defined by either (4.2) or equivalently by (4.9). In terms of the fermion components we have introduced above, this boundary condition can be rewritten as

$$Z^i\mathcal{T}^{i\uparrow} = 0\,, \qquad \bar{Z}^i\bar{\mathcal{T}}^{i\downarrow} = 0\,, \tag{4.22}$$

as well as

$$P^{\bar{j}l}\bar{\mathcal{T}}^{l\uparrow} = 0\,, \qquad P^{\bar{j}l}\mathcal{T}^{j\downarrow} = 0\,. \tag{4.23}$$

Here $P^{\bar{j}l}$ is the projection matrix defined by

$$P^{\bar{j}l} = \left(\delta^{jl} - \frac{\bar{Z}^j Z^l}{|Z|^2}\right). \tag{4.24}$$

For future reference we note that

$$\delta_\uparrow P^{\bar{j}l} = -2\frac{Z^l}{|Z|^2}P^{\bar{j}r}\bar{\mathcal{T}}^{r\downarrow}\,, \qquad \delta_\downarrow P^{\bar{j}l} = -2\frac{\bar{Z}^j}{|Z|^2}P^{\bar{r}l}\mathcal{T}^{r\uparrow}\,. \tag{4.25}$$

It is easy to see that the variations of (4.20) with respect to $\uparrow$ and $\downarrow$ lead to the fermionic boundary conditions (4.22). The variation of these fermionic boundary conditions then either vanish or give

$$Z^i \partial_\tau Z^i = \frac{1}{2}\partial_\tau \left(Z^i Z^i\right) = 0\,, \tag{4.26}$$

as follows from (4.12) and (4.13), or by direct calculation. These two sets of boundary conditions are therefore invariant under the two supersymmetry transformations. This leaves us with analysing the supersymmetry transformations of (4.23). Using (4.25) one calculates

$$\delta_\downarrow \left(P^{\bar{\jmath}l}\bar{\mathcal{T}}^{l\uparrow}\right) = -2\frac{\bar{Z}^j}{|Z|^2}P^{\bar{r}l}\mathcal{T}^{r\uparrow}\bar{\mathcal{T}}^{l\uparrow} - 2i\widehat{m}P^{\bar{\jmath}l}\bar{Z}^l = 0 \tag{4.27}$$

since

$$-\frac{\bar{Z}^j}{|Z|^2}P^{\bar{r}l}\mathcal{T}^{r\uparrow}\bar{\mathcal{T}}^{l\uparrow} = -\frac{\bar{Z}^j}{|Z|^2}\mathcal{T}^{r\uparrow}P^{\bar{r}l}\bar{\mathcal{T}}^{l\uparrow} = 0 \tag{4.28}$$

by (4.23), and

$$P^{\bar{\jmath}l}\bar{Z}^l = \left(\delta^{jl} - \frac{\bar{Z}^j Z^l}{|Z|^2}\right)\bar{Z}^l = \bar{Z}^j - \bar{Z}^j\frac{|Z|^2}{|Z|^2} = 0\,. \tag{4.29}$$

Likewise, one shows that

$$\delta_\uparrow \left(P^{\bar{\jmath}l}\mathcal{T}^{j\downarrow}\right) = 0\,. \tag{4.30}$$

On the other hand, one finds

$$\delta_\uparrow \left(P^{\bar{\jmath}l}\bar{\mathcal{T}}^{l\uparrow}\right) = P^{\bar{\jmath}l}\partial_\sigma Z^l + 2\frac{Z^l\bar{\mathcal{T}}^{l\uparrow}}{|Z|^2}P^{\bar{\jmath}r}\bar{\mathcal{T}}^{r\downarrow} \tag{4.31}$$

and

$$\delta_\downarrow \left(P^{\bar{\jmath}l}\mathcal{T}^{j\downarrow}\right) = P^{\bar{\jmath}l}\partial_\sigma \bar{Z}^j + 2\frac{\bar{Z}^j\mathcal{T}^{j\downarrow}}{|Z|^2}P^{\bar{r}l}\mathcal{T}^{r\uparrow}\,. \tag{4.32}$$

Apart from the bilinear fermion term, the right hand sides of (4.31) and (4.32) are the Neumann boundary conditions one would have expected. Since the bilinear fermion term does not vanish automatically, we are therefore led to postulate that the *actual (Neumann) boundary conditions* for the bosons are given by

$$\begin{aligned} P^{\bar{\jmath}l}\partial_\sigma Z^l + 2\frac{Z^l\bar{\mathcal{T}}^{l\uparrow}}{|Z|^2}P^{\bar{\jmath}r}\bar{\mathcal{T}}^{r\downarrow} &= 0 \\ P^{\bar{\jmath}l}\partial_\sigma \bar{Z}^j + 2\frac{\bar{Z}^j\mathcal{T}^{j\downarrow}}{|Z|^2}P^{\bar{r}l}\mathcal{T}^{r\uparrow} &= 0\,. \end{aligned} \tag{4.33}$$

These two equations are complex conjugates of one another. As we shall see below, they can also be obtained by analysing the variation of the action directly.

Given the supersymmetry relations (4.12) and (4.13), as well as (4.27) and (4.30) it is now easy to see that (4.23) and (4.33) are closed under the different supersymmetry transformations. For example, (4.12) and (4.27) imply that

$$\delta_\uparrow \left(P^{\bar{\jmath}l}\partial_\sigma Z^l + 2\frac{Z^l\bar{\mathcal{T}}^{l\uparrow}}{|Z|^2}P^{\bar{\jmath}r}\bar{\mathcal{T}}^{r\downarrow}\right) = 0\,, \tag{4.34}$$

and in fact this is not difficult to check explicitly. Furthermore, (4.13) and (4.27) imply that

$$\delta_{\downarrow}\left(P^{\bar{j}l}\partial_\sigma Z^l + 2\frac{Z^l\bar{\mathcal{T}}^{l\uparrow}}{|Z|^2}P^{\bar{j}r}\bar{\mathcal{T}}^{r\downarrow}\right) = 2\partial_\tau\left(P^{\bar{j}l}\bar{\mathcal{T}}^{l\uparrow}\right), \tag{4.35}$$

and thus the right hand side vanishes because of (4.23). The other variations are similar.

### 4.3 Varying the action

As mentioned above, the modified bosonic boundary conditions (4.33) can also be obtained directly by varying the action.[11] The boundary terms in the variation of the action $I$ are of the form

$$\delta I_{bound} = \int d\tau\left(\delta X^\mu \frac{\delta I}{\delta(\partial_\sigma X^\mu)} + \delta S\frac{\delta I}{\delta(\partial_\sigma S)} + \delta\tilde{S}\frac{\delta I}{\delta(\partial_\sigma \tilde{S})}\right), \tag{4.36}$$

where $\partial_\tau = \partial_+ + \partial_-$, $\partial_\sigma = \partial_+ - \partial_-$. Using the explicit form of the action (2.1), the last two terms in the integrand are

$$-\frac{1}{2}\left(S\delta S - \tilde{S}\delta\tilde{S}\right). \tag{4.37}$$

This usually vanishes by imposing the boundary condition $S = -M\tilde{S}$, provided that $M$ is an orthogonal constant matrix. In the present context, however, $M$ depends on $Z^i$ and $\bar{Z}^i$, and $S = -M\tilde{S}$ therefore leads to

$$\frac{1}{2}\left[\delta Z^i\left(S\frac{\partial M}{\partial Z^i}\tilde{S}\right) + \delta\bar{Z}^i\left(S\frac{\partial M}{\partial \bar{Z}^i}\tilde{S}\right)\right]. \tag{4.38}$$

On the other hand, the first term in (4.36) leads to

$$-\frac{1}{4}\left(\delta Z^i\,\partial_\sigma\bar{Z}^i + \delta\bar{Z}^i\,\partial_\sigma Z^i\right), \tag{4.39}$$

where $2\delta X^I\partial_\sigma X^I = (\delta Z^i\partial_\sigma\bar{Z}^i + \delta\bar{Z}^i\partial_\sigma Z^i)$ has been used. Combining with (4.38) the variation of $Z^i$ and $\bar{Z}^i$ thus has to satisfy

$$\delta Z^i\left(\frac{1}{2}\partial_\sigma\bar{Z}^i - S\frac{\partial M}{\partial Z^i}\tilde{S}\right) + \delta\bar{Z}^i\left(\frac{1}{2}\partial_\sigma Z^i - S\frac{\partial M}{\partial \bar{Z}^i}\tilde{S}\right) = 0. \tag{4.40}$$

The term proportional to $\delta Z^i$ can be written as

$$\delta Z^i\left(\frac{1}{2}\left[\delta^{ij} - \frac{\bar{Z}^j Z^i}{|Z|^2}\right]\partial_\sigma\bar{Z}^j + \frac{\bar{Z}^j Z^i}{2|Z|^2}\partial_\sigma\bar{Z}^j - S\frac{\partial M}{\partial Z^i}\tilde{S}\right) = 0. \tag{4.41}$$

The second term vanishes since $Z^i Z^i = c$ and the remaining terms combine to give the Neumann conditions

$$\delta Z^i\left(P^{\bar{j}i}\partial_\sigma\bar{Z}^j - 2S\frac{\partial M}{\partial Z^i}\tilde{S}\right) = 0. \tag{4.42}$$

Thus the Neumann conditions get indeed modified by a fermion bilinear term.

[11] Related modifications of the Neumann conditions by fermion-bilinears have recently also been obtained in [16].

One can readily verify that (4.33) satisfies (4.42). In order to see this, one observes that (4.22) implies

$$\frac{\bar{Z}^l \mathcal{T}^{l\downarrow}}{|Z|^2} = 2i \frac{\bar{Z}^k}{|Z|^2} S^{k\downarrow} , \tag{4.43}$$

while (4.23) gives

$$P^{j\bar{r}} \mathcal{T}^{r\uparrow} = 2i P^{j\bar{r}} P^{k\bar{r}} S^{k\uparrow} . \tag{4.44}$$

Putting these together one finds

$$\frac{\bar{Z}^k \mathcal{T}^{k\downarrow}}{|Z|^2} P^{\bar{r}i} \mathcal{T}^{r\uparrow} = -4 \frac{\bar{Z}^k}{|Z|^2} S^{k\downarrow} P^{i\bar{r}} P^{l\bar{r}} S^{l\uparrow} . \tag{4.45}$$

Furthermore $P^{i\bar{r}} P^{l\bar{r}} = P^{l\bar{i}} + Z^i V^l$, and since $Z^i \delta Z^i = 0$, (4.42) equals

$$\delta Z^i \left( 8 \frac{\bar{Z}^k}{|Z|^2} S^{k\downarrow} P^{\bar{i}l} S^{l\uparrow} - 2S \frac{\partial M}{\partial Z^i} \tilde{S} \right) = 0 . \tag{4.46}$$

This then vanishes manifestly since it follows from (4.9) that

$$\frac{\partial M}{\partial Z^i} = 2i \frac{\bar{Z}^j}{|Z|^2} P^{\bar{i}k} (b^\dagger)^k b^j . \tag{4.47}$$

### 4.4 The curved $D5$-brane

The analysis for the curved $D7$-brane can be generalised to other curved branes. For example, the superpotential is constant on the world-volume of a curved $D5$-brane that is characterised by the equations

$$Z_1^2 + Z_2^2 + Z_3^2 = a \tag{4.48}$$

$$Z_4 = b . \tag{4.49}$$

The fermionic gluing matrix, $M_{c5}$, can be taken to be the product of the gamma-matrices in the Dirichlet directions, and is explicitly given by

$$M_{c5} = -i \frac{Z^a \bar{Z}^c}{Z^d \bar{Z}^d} \left[ b^c, (b^a)^\dagger \right] \gamma^4 \gamma^8 . \tag{4.50}$$

Here the indices $a, b, \ldots$ run over the reduced range $1, 2, 3$. On the top and bottom states, $M_{c5} = -\mathbb{1}$, and thus the corresponding dynamical supersymmetries are related as $\epsilon_1 = \epsilon_2$. The analogues of (4.10) is then

$$\delta_\uparrow Z^i = 0 , \qquad \delta_\downarrow \bar{Z}^i = 0 \tag{4.51}$$

$$\delta_\uparrow \bar{Z}^i = \mathcal{J}^{i\downarrow} , \qquad \delta_\downarrow Z^i = \mathcal{J}^{i\uparrow} , \tag{4.52}$$

where the relevant combinations of the fermions are now

$$\mathcal{J}^a = \tilde{S}^a + S^a , \qquad \bar{\mathcal{J}}^a = \tilde{S}^a - S^a . \tag{4.53}$$

Unlike the situation for the $D7$-brane the two expressions are are not complex conjugates of each other. The variations of the fermions can similarly be determined. The bosonic



JHEP10(2003)052

Dirichlet conditions follow directly from (4.48), while the fermionic conditions are again given by $S = -M_{c5}\tilde{S}$. The modified Neumann boundary conditions are in this case

$$\delta_{\uparrow}\left(\hat{P}^{a\bar{b}}\bar{\mathcal{J}}^{a\uparrow}\right) = \hat{P}^{c\bar{b}}\partial_\sigma Z^c + 2\frac{Z^a\bar{\mathcal{J}}^{a\uparrow}}{Z^d Z^d}\,\hat{P}^{e\bar{b}}\bar{\mathcal{J}}^{e\downarrow} = 0\,, \tag{4.54}$$

as well as its complex conjugate, where the modified projector $\hat{P}^{a\bar{b}}$ is defined by

$$\hat{P}^{a\bar{b}} = \delta^{ab} - \frac{Z^a\bar{Z}^b}{Z^d\bar{Z}^d}\,. \tag{4.55}$$

As before, it can be shown that this set of boundary conditions is invariant under the top- and bottom supersymmetry transformation. The bilinear fermionic correction term to the bosonic Neumann condition can also be derived from the action as before.

### 4.5 Curved branes and supersymmetry

The above analysis implies that the curved $D7$-brane preserves two dynamical supersymmetries (namely those associated to the top- and bottom components), but one may wonder whether it preserves any more supersymmetries. In order to analyse this question, it is convenient to study first the corresponding question for the curved $D7$-brane in flat space. The fact that this brane preserves at least two supersymmetries follows directly from the above analysis (since the whole argument goes through for $\widehat{m} = 0$). In fact, the top- and bottom components are the only constant eigenvectors of $M_7$, and one should therefore expect that this brane does not preserve any additional supersymmetries.

Curved supersymmetric branes of this type have been discussed before from the point of view of calibrated geometry, e.g. in [17]. In fact, it is known [18] that any complex submanifold of $\mathbb{C}^n$ is calibrated, and thus allows for covariantly constant spinors. If $F_i(z_1,\ldots,z_n)$, $i = 1,\ldots,l$ is a family of holomorphic functions, then the joint zero locus

$$F_i(z_1,\cdots,z_n) = 0\,, \quad i = 1,\ldots,l \tag{4.56}$$

defines a complex submanifold. For the case of the curved $D7$-brane, the world-volume is the deformed conifold (i.e., the cotangent bundle of $S^3$, $T^*S^3$), which is a Calabi-Yau 3-fold with holonomy group SU(3). The theorem in [19] then implies that there are precisely 2 covariantly constant spinors. Thus the curved $D7$-brane in flat space preserves precisely the two supersymmetries we have found above. In particular, this then implies that the curved $D7$-brane in the pp-wave background also only preserves these two supersymmetries (since the supersymmetry of a brane cannot decrease by taking the limit $\widehat{m} \to 0$).

The analysis for the curved $D5$-brane is similar. By the same arguments as above, the curved $D5$-brane also preserves two dynamical supersymmetries in flat space. As regards the kinematical supersymmetries, $M_{c5}$ acts as $+1$ on both $(+++-)$ and $(---+)$, and therefore the curved $D5$-brane preserves also two kinematical supersymmetries in flat space. (However, since $\Pi M_{c5}\Pi M_{c5} = +1$ on both of these states, these do not give rise to kinematical supersymmetries in the pp-wave background.) This is consistent with the argument based on calibrated geometries. For the curved $D5$-brane the world-volume is the cotangent bundle of $S^2$, $T^*S^2$. This defines a Calabi-Yau 2-fold with holonomy group

SU(2). The theorem in [19] then implies again that there are two dynamical supersymmetries. The additional kinematical supersymmetries arise from the additional two transverse flat directions parametrised by $z_4 = c$.

The supersymmetry of the curved $D7$-brane is also consistent with the following geometric picture of its world-volume: it can be thought of as being a ruled surface, where the lines are given by the world-volume of oblique $D3$-branes, which sweep out the $D7$-brane under the action of the diagonal SO(4) subgroup of the SO(4)×SO(4) isometry of the background. More explicitly, consider the oblique D3-brane whose world-volume is described by (1.6) with $a = c^{1/2}$ and $b = 0$. Then this D3-brane lies inside the world-volume of the curved D7-brane that is described by the equation

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = c\,. \tag{4.57}$$

Furthermore, any point on (4.57) is related by an element in the diagonal SO(4) subgroup to a point on the above oblique $D3$-brane. Thus the oblique $D3$-brane sweeps out the full world-volume of the curved $D7$-brane under this SO(4) action. From the analysis in [20] it follows that the different $D3$-branes, that are related by an SO(4) rotation in this manner, preserve two common supersymmetries, which are precisely the top and bottom spinors in our context.

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